

AC 2007-3116: RULE-BASED MODELING OF DYNAMIC SYSTEMS AND COMPUTATION OF TRANSFER-FUNCTIONS WITH "MATLAB"

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RULE BASED MODELING OF DYNAMIC SYSTEMS AND COMPUTATION OF TRANSFER-FUNCTIONS WITH “MATLAB”

Abstract

The dynamic systems are in general modeled by Newton's force method or Lagrange's energy method. In this paper a rule based procedure is given as an alternate method for simple linear systems. The resulting equations in matrix form can be used as input in the analytical computation of transfer functions with MATLAB software. Integration of rule based modeling of dynamic systems and analytical computation of transfer functions with MATLAB provides a viable pedagogical tool for undergraduate education in Linear System Control.

Introduction

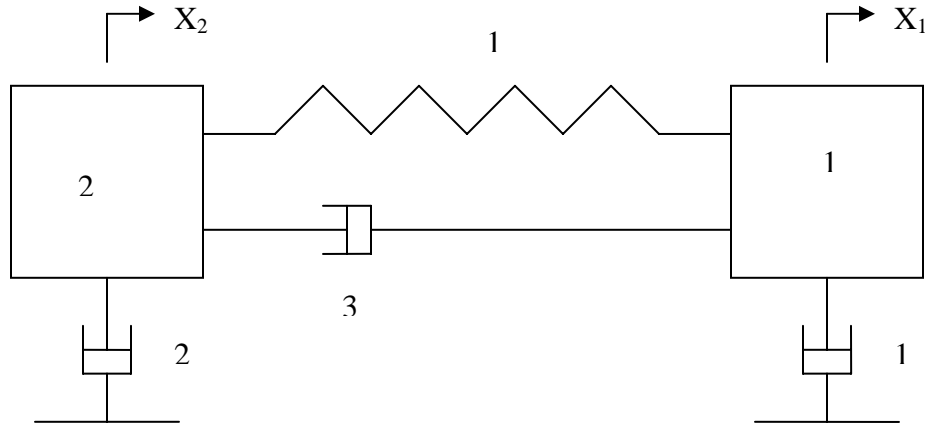
Linear Vibrations and Linear System Control are relatively difficult courses in undergraduate curriculum for Aerospace, Mechanical as well as Electrical engineering students. The primary difficulty lies in developing the equations of motion from given physical system. In applying the Newton's force method, one should have good understanding of degree of freedom and free body diagram. In applying the energy method, one should have good understanding of kinetic energy, potential energy, and damping energy. Although some students have very good grasp of these principles, majority of the class have trouble in application of these concepts. The computation of transfer functions in analytical form requires the knowledge of partial fractions and Laplace Transforms. Manual computation of transfer functions is rather laborious for even some simple systems. MATLAB software can be easily used for this purpose and it is efficient in applying Laplace Transform in order to find the transfer function.

Rules for Modeling

1. *Degrees of Freedom:* (a) Every degree of freedom $[X_i]$ is associated with a mass $[M_i]$; (b) A degree of freedom without a real mass is associated with a virtual mass of zero magnitude (or zero-mass).
2. *Stiffness Matrix:* (a) Diagonal elements of the matrix $[K_{ii}]$ are sum of the values of stiffness connected to each mass $[M_i]$, including the zero-mass. (b) The Off-diagonal terms $[K_{ij}]$ are negative and equal to sum of the values of stiffness connected in between mass $[M_i]$ and $[M_j]$.
3. *Damping Matrix:* (a) Diagonal elements of the matrix $[C_{ii}]$ are sum of the values of damping connected to each mass $[M_i]$, including the zero-mass. (b) The Off-diagonal terms $[C_{ij}]$ are negative and equal to sum of the values of damping connected in between mass $[M_i]$ and $[M_j]$.

4. *Mass Matrix*: (a) Diagonal elements of the matrix $[M_{ij}]$ are the value of mass $[M_i]$ that correspond to the degree of freedom $[X_i]$. However, the value of the zero-mass is zero. (b) The Off-diagonal terms $[M_{ij}]$ are zero.

Example 1:



In this example, there are two masses connected to a spring and three viscous dampers with two degrees of freedom. The values indicated at the mass, spring and damper represent the magnitude of each in their appropriate units. Now the matrices $[M]$, $[C]$, and $[K]$ can be developed by applying the rules as shown below.

Rule #1: Degrees of Freedom

This system has two degrees of freedom (X_1, X_2) and each one is associated with a mass and the masses are real.

Rule #2: Stiffness Matrix

Diagonal element $K_{11} = 1$ (the value of all the stiffness connected to the mass with X_1 as degree of freedom)

Diagonal element $K_{22} = 1$ (the value of all the stiffness connected to the mass with X_2 as degree of freedom)

Off-diagonal element $K_{12} = -1$ (the value of all the stiffness connected between the two masses)

Off-diagonal element $K_{21} = K_{12} = -1$ (symmetry of the matrix)

Rule #3: Damping Matrix

Diagonal element $C_{11} = 4$ (the value of all the damping connected to the mass with X_1 as degree of freedom)

Diagonal element $C_{22} = 5$ (the value of all the damping connected to the mass with X_2 as degree of freedom)

Off-diagonal element $C_{12} = -3$ (the value of all the damping connected between the two masses)

Off-diagonal element $C_{21} = C_{12} = -3$ (symmetry of the matrix)

Rule #4: Mass Matrix

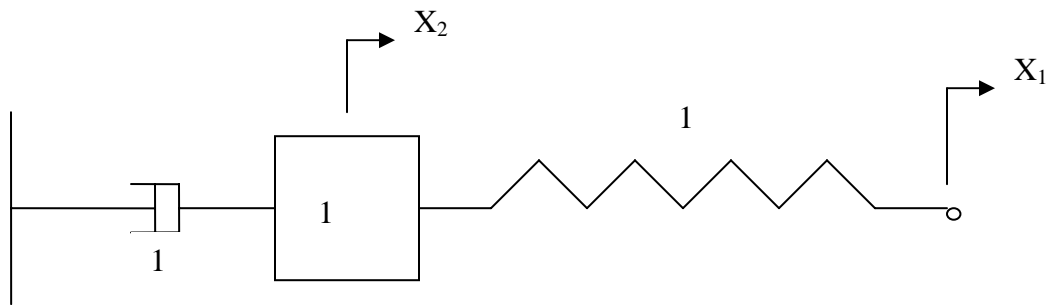
Diagonal elements $M_{11} = 1$ and $M_{22} = 2$

Off-diagonal elements $M_{12} = M_{21} = 0$

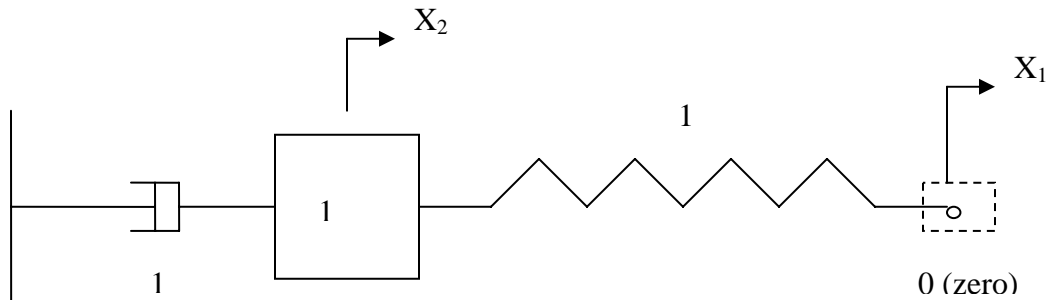
Now the mass, damping and stiffness matrices can be written as,

$$[M] = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}; \quad [C] = \begin{pmatrix} 4 & -3 \\ -3 & 5 \end{pmatrix}; \quad [K] = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

Example 2:



In this example, there is only one mass connected to a spring and damper with two degrees of freedom. While the Rule #1 requires that every degree of freedom is associated with a mass, the degree of freedom X_1 is not associated with a mass. So, a virtual mass of zero magnitude (zero-mass) is introduced at the degree of freedom X_1 as shown below.



Rule #1: Degrees of Freedom

This system has two degrees of freedom (X_1, X_2). The degree of freedom X_1 is associated with a virtual mass while the degree of freedom X_2 is associated with a real mass. The virtual mass is shown with dashed line and its magnitude is zero.

Rule #2: Stiffness Matrix

Diagonal element $K_{11} = 1$ (the value of all the stiffness connected to the mass with X_1 as degree of freedom)

Diagonal element $K_{22} = 1$ (the value of all the stiffness connected to the mass with X_2 as degree of freedom)

Off-diagonal element $K_{12} = -1$ (the value of all the stiffness connected between the two masses)

Off-diagonal element $K_{21} = K_{12} = -1$ (symmetry of the matrix)

Rule #3: Damping Matrix

Diagonal element $C_{11} = 0$ (the value of all the damping connected to the mass with X_1 as degree of freedom)

Diagonal element $C_{22} = 1$ (the value of all the damping connected to the mass with X_2 as degree of freedom)

Off-diagonal element $C_{12} = 0$ (the value of all the damping connected between the two masses)

Off-diagonal element $C_{21} = C_{12} = 0$ (symmetry of the matrix)

Rule #4: Mass Matrix

Diagonal elements $M_{11} = 0$ (zero-mass) and $M_{22} = 1$

Off-diagonal elements $M_{12} = M_{21} = 0$

Now the mass, damping and stiffness matrices can be written as,

$$[M] = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}; [C] = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}; [K] = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

General Solution

The general equation of motion for a given system can be written as,

$$[M]\{\ddot{x}\} + [C]\{\dot{x}\} + [K]\{x\} = \{F(t)\}$$

Let $x(0)$ and $\dot{x}(0)$ be the initial conditions representing the initial displacement and initial velocity respectively. Applying the Laplace Transform, the above equation can be written as,

$$\langle s^2 [M] + s[C] + [K] \rangle \{x(s)\} = \{F(s)\} + \langle s[M] + [C] \rangle \{x(0)\} + [M]\{\dot{x}(0)\}$$

For zero initial conditions,

$$\{x(s)\} = [H(s)]\{F(s)\}; \text{ where the Transfer Function, } [H(s)] = \langle s^2[M] + s[C] + [K] \rangle^{-1}$$

This matrix equation for the transfer function can be evaluated in analytical form using the software, "MATLAB". In general for multi degree of freedom, the transfer function can be expressed as,

$$H_{ij} = \frac{x_i(s)}{F_j(s)}$$

For example, if a system has two degrees of freedom with forcing function acting at each degree of freedom, the forcing function need to be applied one at a time in the evaluation of the transfer functions. Assuming $F_1(t)$ is the only forcing function acting on the system and $F_2(t)=0$,

$$H_{11} = \frac{x_1(s)}{F_1(s)}; H_{21} = \frac{x_2(s)}{F_1(s)}$$

Assuming $F_2(t)$ is the only forcing function acting on the system and $F_1(t)=0$,

$$H_{12} = \frac{x_1(s)}{F_2(s)}; H_{22} = \frac{x_2(s)}{F_2(s)}$$

Computation and Results

This procedure can be easily programmed in "MATLAB" for finding the transfer function in closed form and the code is given below.

MATLAB Code

```
clc, clear all
syms s dmat;
fprintf('Enter the parameters of the system\n\n')
n = input('order of the system := ');
m = input('M := ');
c = input('C := ');
k = input('K := ');
for i = 1:n
    for j = 1:n
        dmat(i,j) = m(i,j) * s^2 + c(i,j)*s + k(i,j);
    end
end
end
h = inv(dmat);
fprintf('\n Transfer Funtion: \n')
transfn(h)
```

Example-1:

The MATLAB template for Example-1 is given below:

```

Command Window
Enter the parameters of the Mechanical system

Order of the system := 2
M := [1 0;0 2]
C := [4 -3;-3 5]
K := [1 -1;-1 1]

Transfer Function:

      2
      [ 2 s + 5 s + 1      3 s + 1 ]
      [-----]
      [      s %1          s %1 ]
      [ ]
      [ ]
      [      2 ]
      [ 3 s + 1      s + 4 s + 1 ]
      [-----]
      [      s %1          s %1 ]

      3      2
      %1 := 2 s + 13 s + 14 s + 3

>>

```

It can also be written as,

$$H(s) = \frac{1}{\Delta} \begin{pmatrix} (2s^2 + 5s + 1) & (3s + 1) \\ (3s + 1) & (s^2 + 4s + 1) \end{pmatrix}; \Delta = s(2s^3 + 13s^2 + 14s + 3)$$

Example-2

The results for Example-2 can be summarized as,

$$H(s) = \frac{1}{\Delta} \begin{pmatrix} (s^2 + s + 1) & 1 \\ 1 & 1 \end{pmatrix}; \Delta = s(s + 1)$$

The MATLAB template of computation is given below.

```

Command Window
Enter the parameters of the Mechanical system

Order of the system := 2
M := [0 0;0 1]
C := [0 0;0 1]
K := [1 -1;-1 1]

Transfer Function:

      [ 2 ]
      [ s + s + 1      1 ]
      [-----]
      [ s (s + 1)      s (s + 1) ]
      [ ]
      [ 1      1 ]
      [-----]
      [ s (s + 1)      s (s + 1) ]

>>

```

Since the transfer function is computed in closed form, they can be readily used for further analysis in linear system control or in stability and control of aerospace vehicles.

Conclusions

Finding the transfer functions for a given dynamic system has been a difficult endeavor for most of the students in Aerospace, Mechanical as well as Electrical engineering programs. Rule based approach to develop the required mass, damping and stiffness matrices directly from the dynamic system eliminated the need for cumbersome procedure of deriving the equations of motion from first principles and then reducing them to required matrices. This approach is very well received by students and it is not in general available in existing text books. Applying the Laplace Transform to matrix equations of motion and extracting the transfer function by solving a series of partial fractions has been eliminated by using the MATLAB software. Since MATLAB is commonly used by most of the engineering students it would be appropriate to use for this purpose.

Integrating the rule based approach and the application of MATLAB not only simplifies the process of obtaining the transfer functions, and also provides a valuable pedagogical tool to educate the engineering leaders of tomorrow.

Authors' Biography

Dr. Swami Karunamoorthy is the professor of Aerospace and Mechanical Engineering at Saint Louis University. He is an Associate Fellow of AIAA, Fellow of ASME, and an active member of ASEE, AHS. He is also a Commissioner and Program Evaluator for ABET. His areas of interest are Engineering Education, Assessment, Vibrations, Linear System Controls, Composites, Helicopters, Aerospace Structures and Applied Aerodynamics. He has several publications on these areas.

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