

AC 2007-418: A LIBRARY OF MATLAB SCRIPTS FOR ILLUSTRATION AND ANIMATION OF SOLUTIONS TO PARTIAL DIFFERENTIAL EQUATIONS

Raymond Jacquot, University of Wyoming

Ray Jacquot, Ph.D., P.E., received his BSME and MSME degrees at the University of Wyoming in 1960 and 1962 respectively. He was an NSF Science Faculty Fellow at Purdue University where he received the Ph.D. in 1969. He joined the Electrical Engineering faculty of the University of Wyoming in 1969. He is a member of ASEE, IEEE and ASME and has been active in ASEE for over three decades serving as Rocky Mountain Section Chair and PIC IV Chair. His professional interests are in modeling, control and simulation of dynamic systems. He is currently Professor Emeritus of Electrical and Computer Engineering. E-mail: quot@uwyo.edu.

Cameron Wright, University of Wyoming

Cameron H. G. Wright, Ph.D, P.E., is on the faculty of the Department of Electrical and Computer Engineering at the University of Wyoming, Laramie, WY. He was previously Professor and Deputy Department Head of the Electrical Engineering Department at the U.S. Air Force Academy. His research interests include signal and image processing, biomedical instrumentation, communications systems, and laser/electro-optics applications. Dr. Wright is a member of ASEE, IEEE, SPIE, NSPE, Tau Beta Pi, and Eta Kappa Nu. E-mail: c.h.g.wright@ieee.org.

Robert Kubichek, University of Wyoming

Robert F. Kubichek received his Ph.D. from the University of Wyoming in 1985. He has worked in research positions at the BDM Corporation and the Institute for Telecommunication Sciences (U.S. Dept. of Commerce), and was an adjunct professor at the University of Colorado from 1989-1991. He joined the University of Wyoming in 1991, where he is currently an Associate Professor. Current research interests include speech analysis for intelligibility and speech quality, and developing new diagnostic tools for speech disorders. E-mail: kubichek@uwyo.edu.

Thomas Edgar, University of Wyoming

Thomas Edgar received the Ph.D. from Colorado State University in 1983. He teaches geotechnical engineering and groundwater hydrology courses in the Department of Civil and Architectural Engineering at the University of Wyoming. He is an Associate Professor and has been an award winning teacher in University and the Department. He is currently the coordinator for the freshman orientation classes in the college. E-mail tvedgar@uwyo.edu.

A Library of MATLAB™ Scripts for Illustration and Animation of Solutions to Partial Differential Equations

Introduction

In the past three years the authors have developed a series of MATLAB™ scripts that illustrate the solutions to partial differential equations commonly encountered in mathematics, engineering and physics courses. The objective of this paper is to create awareness, among teaching faculty, of the availability of this set of MATLAB™ scripts to aid their teaching of physical phenomena governed by partial differential equations.

Over many years the authors have observed the difficulty students have with the solutions to partial differential equation problems and when they have completed such a solution they still cannot associate a physical interpretation with the resulting equation or equations. Since many students are graphical learners, we asked ourselves how the high quality and easy to use graphics available in MATLAB™ might be exploited to help students better understand the solutions that they, their instructor or the textbook may have generated.

There has been considerable work done to exploit the use of computer graphics to clarify physical problems governed by partial differential equations. An early paper used MATLAB™ to illustrate solutions to hyperbolic differential equations.¹ Several papers at about the same time used computer animation to illustrate solutions for elastic wave propagation and beam vibration.^{2,3} The concept of using MATLAB™ for the animation of lumped parameter dynamic systems was demonstrated by Watkins et al.⁴ Recently there have been a number of papers describing the graphical interpretation of partial differential equations. The transport of pollutants in groundwater has been described using web-based graphics⁵ and another paper reports a virtual laboratory for teaching quasistationary electromagnetics.⁶ Another recent paper discusses the solution of groundwater problems using a spreadsheet.⁷ Still another paper employs a spreadsheet to examine the topic of electromagnetic wave propagation.⁸ Two recent papers reported the use of animation to clarify a variety of partial differential equation solutions.^{9,10} There are a number of approaches to the animation of distributed parameter systems and one is the application of finite element software (ANSYS™) to illustrate the vibration of beams and plates.¹¹ A recent paper discusses the use of animation in MATLAB™ to animate the solution to a variety of electrical transmission line problems.¹² A very recent paper discusses how MATLAB™ has been employed to illustrate the downwind transport of the chemical components of industrial stack emissions.¹³

Strategies for PDE Solution Presentation

There are a number of possible ways that graphical presentations may be employed to clarify responses of dynamic systems described by partial differential equations with one spatial variable and time as independent variables. The most obvious are:

- A plot of the solution as a function of time for several locations (location as a parameter).
- A plot of the solution as a function of the spatial variable for several values of time (time as a parameter).

- A 3-d plot of the solution as a function of location and time.
- An animation of the solution as a function of the spatial variable as time evolves. This is a closely spaced (in time) version of the second method.

Although when the project was initiated it was thought that a 3-d image of the solution might be superior, the authors have discovered that by far the most effective of the above-mentioned presentation schemes is the one involving animation.

MATLAB scripts for a variety of physical problems involving one spatial variable and time have been written. The exception involves the steady flow streamlines for a fluid dynamics problems.

In that which follows the problems are categorized by the type of physical problem solved and the number of scripts for each is given below.

- Electrical transmission lines—5 scripts
- Beam vibration—9 scripts
- Heat conduction—15 scripts
- Beach nourishment—1 script
- String vibration—4 scripts
- Groundwater drawdown—3 scripts
- Wave propagation in elastic bars—6 scripts
- Static beam bending problems—3 scripts
- Fluid dynamics—1 script
- General mathematics—2 scripts

We shall now present several examples that have not been published previously in order to illustrate to the reader what the scripts do.

Example 1--Impulsively Driven Cantilever Beam

Consider the situation illustrated in Figure 1 showing a Bernoulli-Euler cantilever beam of length L with bending stiffness EI and mass per unit length μ driven by an impulse of intensity I_0 at location $x = a$.

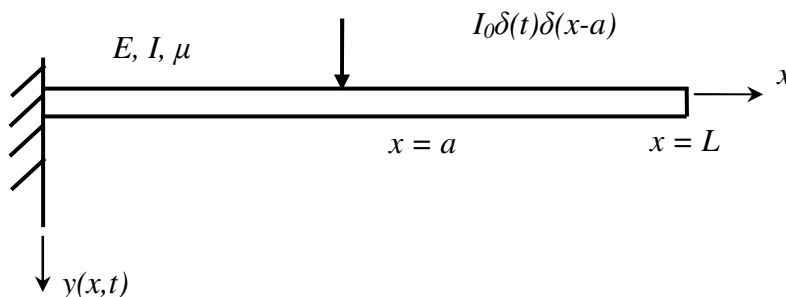


Figure 1. Bernoulli-Euler Cantilever Beam Driven by an Impulse of Strength I_0 .

The beam is governed by the Bernoulli-Euler beam equation

$$\mu \frac{\partial^2 y}{\partial t^2} + EI \frac{\partial^4 y}{\partial x^4} = I_0 \delta(t) \delta(x - a) \quad (1)$$

The appropriate boundary conditions are

$$y(0, t) = \frac{\partial y(0, t)}{\partial x} = EI \frac{\partial^2 y(L, t)}{\partial x^2} = EI \frac{\partial^3 y(L, t)}{\partial x^3} = 0 \quad (2)$$

For these boundary conditions the eigenfunctions (normal modes) are the well-known beam functions

$$\varphi_i(x) = \cosh \beta_i x - \cos \beta_i x - \alpha_i (\sinh \beta_i x - \sin \beta_i x) \quad i = 1, 2, 3, \dots \quad (3)$$

where the values of α_i and $\beta_i L$ are tabulated in Table 1.

Mode i	$\beta_i L$	α_i
1	1.8751	0.7340
2	4.6940	1.0184
3	7.8547	0.9992
4	10.9955	1.0000
5	14.1371	0.9999

The solution to this problem may be given in terms of a generalized Fourier series in the beam functions with time varying coefficients. The resulting solution for zero initial deflection and velocity is

$$y(x, t) = \frac{I_0}{\mu L} \sum_{i=1}^{\infty} \frac{1}{\omega_i} \sin(\omega_i t) \varphi_i(a) \varphi_i(x) \quad (4)$$

where the i th radian natural frequency is defined as

$$\omega_i = \frac{(\beta_i L)^2}{L^2} \sqrt{\frac{EI}{\mu}} \quad i = 1, 2, 3, \dots \quad (5)$$

In this presentation it will be assumed that the impulse is applied at the free end ($a = L$) although in the software the spatial location is an input quantity. The temporal responses at five locations along the beam are illustrated in Figure 2.

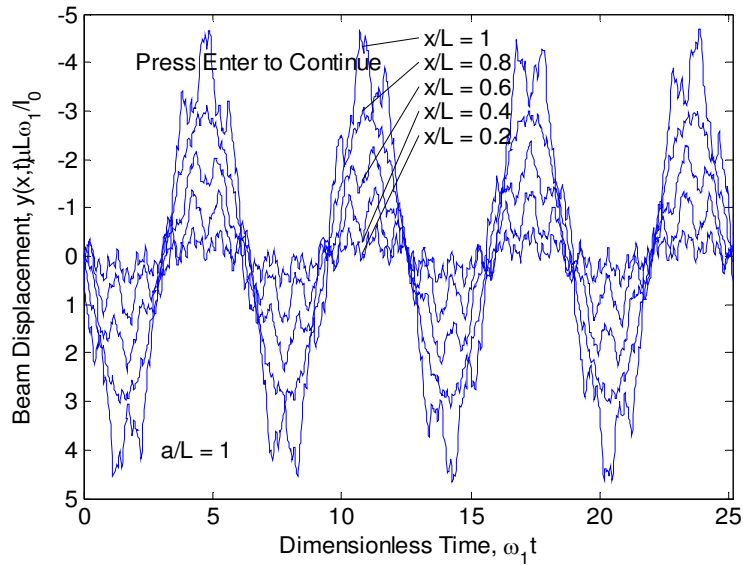


Figure 2. Elastic Cantilever Beam Responses at Various Locations to a Tip Impulse.

A second way to present the response data is to plot the responses as functions of location for a series of times. Figure 3 illustrates the beam responses for twenty equally spaced times over a time interval equal to the first natural period. The software also animates this presentation. Note that the irregular nature of the shapes in Figures 2 and 3 are due to the higher natural frequencies not being integer multiples of the fundamental frequency and hence the motion is not periodic.

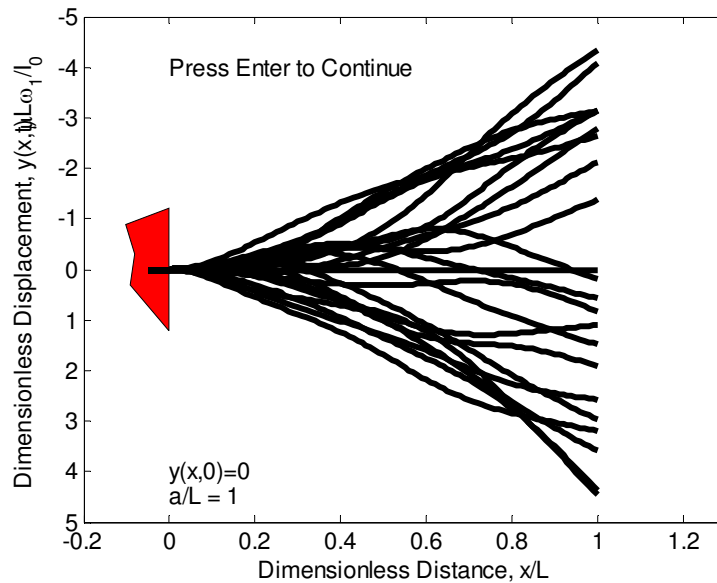


Figure 3. Samples of the Response at Various Times for One Fundamental Natural Period.

Example 2--Blasiuus Boundary Layer Model

Recently Naraghi demonstrated the use of Excel to compute and illustrate the solution to the laminar boundary layer problem over a flat plate.¹⁴ This paper solved both the fluid mechanical and thermal boundary layer problems but did not examine in detail the velocity distributions within the boundary layer. For a complete explanation of the problem the text of Schlichting¹⁵ is an excellent reference. The authors thought that an interesting extension of that work would be the animation of the streamlines for the fluid mechanical boundary layer. The situation to be considered here is illustrated in Figure 4.

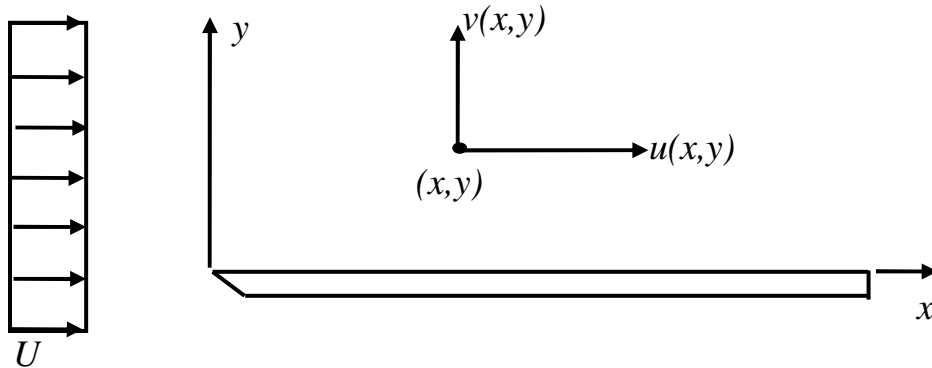


Figure 4. Fluid Flow over a Flat Plate.

The fluid dynamics are governed by the momentum balance in the x -direction which for steady flow and no pressure gradient or body forces is

$$\frac{\mu}{\rho} \frac{\partial^2 u}{\partial y^2} = u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \quad (6)$$

where u and v are respectively the x and y components of the velocity field. ρ and μ are respectively the density and the dynamic viscosity of the fluid. The continuity equation for incompressible flow in two dimensions is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (7)$$

The associated boundary conditions are

$$u(x, 0) = 0, \quad v(x, 0) = 0, \quad u(x, \infty) = U \quad (8)$$

The problem is solved by construction of a stream function

$$\psi = \left[\frac{U\mu x}{\rho} \right]^{\frac{1}{2}} f(\eta) \quad (9)$$

where the new independent similarity transformation variable η is defined as

$$\eta = \left[\frac{\rho U}{\mu x} \right]^{\frac{1}{2}} y \quad (10)$$

The velocities are given by the respective derivatives of the stream function

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x} \quad (11)$$

If the appropriate derivatives of the stream function are substituted for the velocities in the momentum equation the result is a simple nonlinear ordinary differential equation, the Blasius equation

$$2f''' + f'f = 0 \quad (12)$$

where the prime denotes the derivative with respect to the similarity transformation variable η . The boundary conditions (8) dictate the boundary conditions on $f(\eta)$ or

$$f(0) = 0, \quad f'(0) = 0, \quad f'(\infty) = 1 \quad (13)$$

This is a two point boundary value problem and it may be solved numerically by estimating $f''(0)$ and solving the equation until a steady solution for f'' is reached then re-estimating $f''(0)$ and solving again until the final condition on f'' is satisfied. The solution and the first derivative are illustrated in Figure 5.

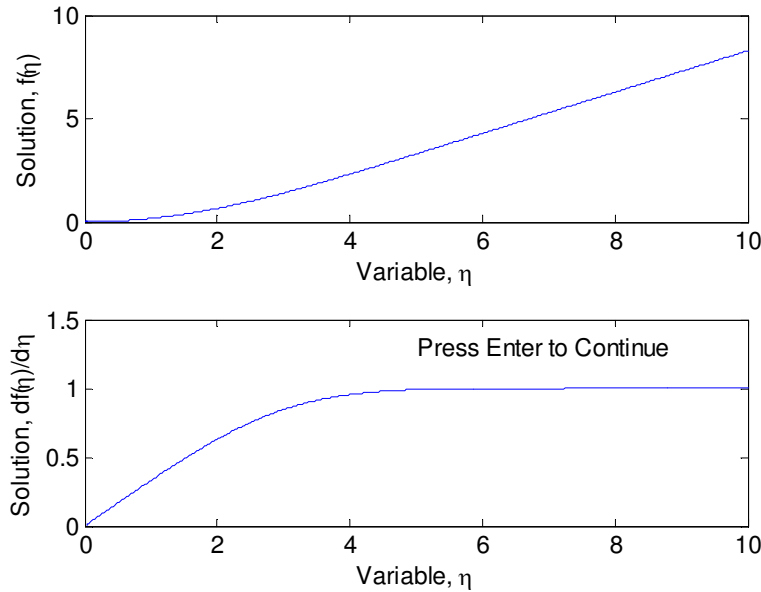


Figure 5. The Solution to the Blasius Equation and the First Derivative Thereof.

Once the solution $f(\eta)$ and its derivative are known then the velocity components at location (x,y) are

$$u(x, y) = Uf'(\eta), \quad v(x, y) = U \left[\frac{\mu}{4\rho Ux} \right]^{\frac{1}{2}} [\eta f'(\eta) - f(\eta)] \quad (14)$$

where η is defined in relation (10). Velocity profiles for various locations x are illustrated in Figure 6 showing the development of the boundary layer from the uniform flow for variables $\rho U/\mu = 1 \times 10^5 \text{ m}^{-1}$ and $U = 0.1 \text{ m/s}$. The boundary layer thickness δ is the locus of points where the horizontal velocity is 99% of the freestream velocity U and is

$$\delta = 5 \sqrt{\frac{\mu x}{\rho U}} \quad (15)$$

The velocity field streamlines and boundary layer thickness are illustrated in Figure 7 for the above-stated variables and $\Delta t = 5 \times 10^{-5}$ s.

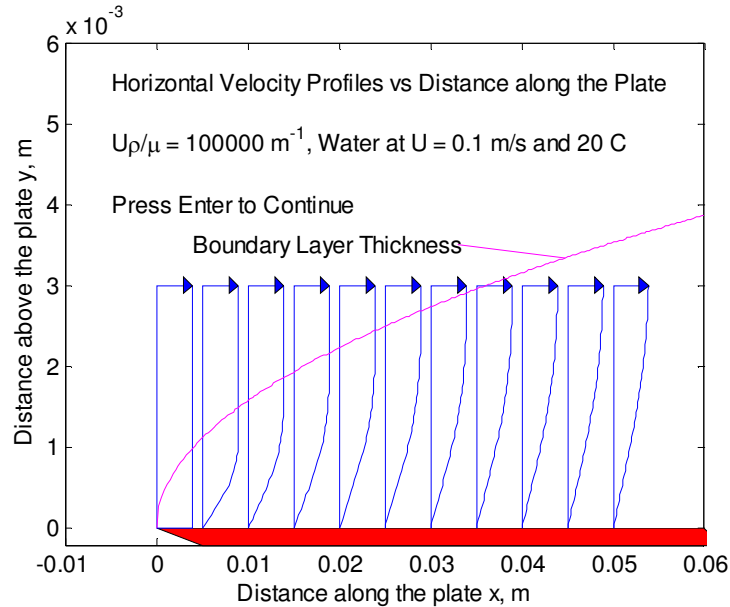


Figure 6. Boundary Layer Development for Laminar Flow.

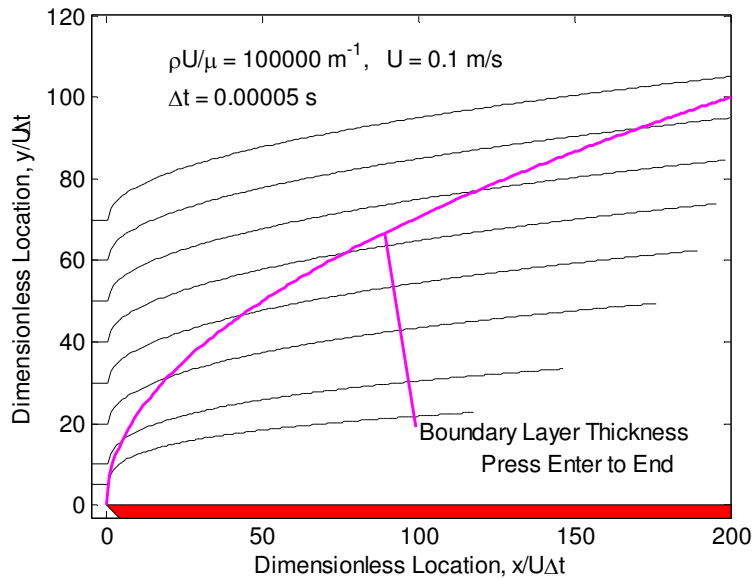


Figure 7. Streamlines and Boundary Layer Thickness for the Blasius Model.

In Figure 7 each streamline is drawn for an equal time duration. It is clear that the velocities nearer the plate surface are lower than those further away. The continuity equation also tells us that when the horizontal velocity decreases the vertical velocity must increase.

Conclusion

The authors' attempts to animate the solution to problems with two spatial variables and time revealed that the time to render the 3-d images in MATLAB™ is excessive and hence this is a strategy awaiting a new generation of hardware and software.

The scripts developed should be useful to teachers in engineering disciplines, physics and mathematics and are available without charge at the authors' website. Appendix A of this paper gives the title and a short description of the problem solved by each script all of which are available for download from the authors' website:

<http://www.eng.uwyo.edu/classes/matlabanimate>

A measure of success of this project will be a monitoring of the number of hits to the website and the time spent at the website.

References

1. J.H. Matthews, Using MATLAB to Obtain Both Numerical and Graphical Solutions to Hyperbolic PDEs, *Computers in Education Journal*, vol. 4, no. 1, Jan./Mar., 1994, pp. 58-60.
2. I. Yusef, K. Slater and K. Gramoll, Using 'GT Vibrations' in Systems Dynamics Courses, *Proc. 1994 ASEE Annual Conference*, June 26-29, Edmonton, Alberta Canada, pp. 952-958. Visualization using Longitudinal
3. K. Slater, and K. Gramoll, Vibration Visualization using Longitudinal Vibration Simulator (LVS), *Proc. 1995 ASEE Annual Conference*, June 25-29, Anaheim, CA, pp.2779-2783.
4. J. Watkins, G. Piper, K. Wedeward and E.E. Mitchell, Computer Animation: A Visualization Tool for Dynamic Systems Simulations, *Proc. 1997 ASEE Annual Conference*, June 15-18, 1997, Milwaukee, WI, Paper 1620-4.
5. A. J. Valocchi and C.J. Werth, Web-Based Interactive Simulation of Groundwater Pollutant Fate and Transport, *Computer Applications in Engineering Education*, vol. 12, no. 2, 2004, pp.75-83.
6. M. de Magistris, A MATLAB Based Virtual Laboratory for Teaching Quasi-Stationary Electromagnetics, *IEEE Transactions on Education*, vol. 48, no. 1, Feb. 2005, pp.81-88.
7. H. Karahan and M. T. Ayvaz, Time Dependent Groundwater Modelling Using Spreadsheet, *Computer Applications in Engineering Education*, vol. 13, no. 3, 2005, pp.192-199.
8. D.W. Ward and K.A. Nelson, Finite-Difference Time-Domain (FDTD) Simulations of Electromagnetic Wave Propagation Using a Spreadsheet, *Computer Applications in Engineering Education*, vol. 13, no. 3, 2005, pp.213-221.
9. R.G. Jacquot, C.H.G. Wright, T.V. Edgar and R.F. Kubichek, Clarification of Partial Differential Solutions Using 2-D and 3-D Graphics and Animation, *Proc. 2005 ASEE Annual Conference and Exposition*, Portland, OR, June 12-15, 2005, Paper 1320-2.
10. R.G. Jacquot, C.H.G. Wright, T.V. Edgar and R.F. Kubichek, Visualization of Partial Differential Equation Solutions, *Computing in Science and Engineering*, vol. 8, no. 1, January/February 2006, pp.73-77.
11. J.R. Barker, ANSYS Macros for Illustrating Concepts in Mechanical Engineering Courses, *Proc. 2005 ASEE Annual Conference and Exposition*, Portland, OR, June 12-15, 2005, Paper 1320-5.
12. R. G. Jacquot, C.H.G. Wright and R.F. Kubichek, Animation Software for Teaching Electrical Transmission Lines, *Proc. 2006 ASEE Annual Conference and Exposition*, Chicago IL, June 18-21, 2006, Paper 1120-1
13. E. Fatehifar, A. Elkamel and M. Taheri, A MATLAB-based Modeling and Simulation Program for Dispersion of Multipollutants from an Industrial Stack for Educational Use in a Course on Air Pollution Control, *Computer Applications in Engineering Education*, vol. 14, no. 4, 2006, pp.300-312.
14. M.N. Naraghi, Solution of Similarity Transform Equations for Boundary Layers Using Spreadsheets, *Computers in Education Journal*, vol. 14, no. 4, Oct./Dec., 2004, pp. 62-69.
15. H. Schlichting, *Boundary Layer Theory*, 7th. Ed. McGraw-Hill, New York, 1979.

Appendix A

The following is a listing of the MATLAB™ scripts, listed by general application category which corresponds to the categories listed early in the paper.

Electrical Transmission Lines

- tls.m Displays solution to lossless, sinusoidally driven transmission line, has GUI. Requires MATLAB version 7.0 or newer.
- tls.png Contains a drawing used by tls.m
- tls.fig Contains the graphics for the GUI used by tls.m.
- transmline2.m Displays solution to the lossless, sinusoidally driven transmission line the same as tls.m. Does not have a GUI and user must change parameters in the script. Specific source and line parameters are specified in the script and the input is the load impedance Z_L .
- lossytransmline.m Displays solution to a lossy, sinusoidally driven transmission line. Specific source and line parameters are specified in the script. Input is the load impedance Z_L .
- transmwave3.m Displays the solution to lossless line driven by a d.c. source. Specific source and line parameters are specified in the script and the load resistance R_L is an input.
- transmlinepulse.m Displays the solution to lossless line driven by a rectangular pulse. Source and line parameters are specified in the script and the pulse width and the load resistance R_L is an input.

Beam Vibration:

- beamvibration.m Displays the solution to a free vibration of a cantilever beam from an initial displacement. Uses generalized Fourier series in the orthogonal beam functions. The initial deflection shape is $y(x,0) = y_0[0.667(x/L)^2 + 0.333(x/L)^3]$.
- cantvib2.m Solves the same problem as beamvibration.m except the beam is discretized spatially using 8 nodes and finite differences in space.
- clampedclampedbeam.m Displays the free vibration solution to a clamped-clamped beam starting with an initial condition $y(x,0) = 2(x/L)^2 - (x/L)^3 - 4(x/L)^4 + 3(x/L)^5$.
- cantbeamimpulse.m Displays the response of a cantilever beam driven by an impulse function of intensity I_0 at a location $x = a$. Input quantity is a/L .
- forcedbeamvibration.m Displays the vibration of a cantilever beam driven by a uniform distributed force f_0 which is constant in time and suddenly applied at $t = 0$.
- cantbeamanimation.m Displays the motion of a cantilever beam excited by a sinusoidal displacement of amplitude Y_0 at the fixed end.
- canttipforceanimation.m Displays the steady-state sinusoidal vibration of a cantilever beam forced at the free end with a sinusoidal force of amplitude F_0 .
- movingload2.m Displays the motion of a simply supported beam with a moving load P starting from the left end with a user controlled velocity. The input variable is the ratio of the transit time to the first natural period of the beam.

ssbeamdispex.m Displays the motion of a simply supported beam driven by a sinusoidal displacement of amplitude Y_0 at the left end.

Heat Conduction:

- conduction.m Displays solution to the diffusion equation for $T=T_0$ at left boundary and $T=0$ at right boundary, zero initial temperature, Fourier series solution.
- conduction2.m Displays solution to the diffusion equation for $T=T_0$ at left boundary and $T=T_0$ at right boundary, zero initial temperature, Fourier series solution.
- conduction3.m Displays solution to the diffusion equation for $T=T_0$ at left boundary and $T=0$ at right boundary, zero initial temperature-- finite difference solution.
- conduction4.m Displays solution to the diffusion equation with convective boundaries for $T = T_0$ at left boundary and $T = 0$ at right boundary, zero initial temperature. This is a finite difference solution in space.
- conduction5.m Displays solution to the diffusion equation for $T = 0$ at left boundary and $T = 0$ at right boundary, T_0 initial temperature, Fourier series solution.
- conduction6.m Displays temperature distribution in a slab with both faces insulated and initially the left half at T_0 and the right half at zero temperature.
- infiniteslab.m Displays the temperature in an infinite slab with initial temperature T_0 between $-L$ and L at $t = 0$ and zero elsewhere.
- convboundaries.m Displays temperatures in a finite slab with convective heat transfer coefficients h_1 and h_2 on the left and right boundaries respectively. The film coefficients are assumed to be the same on both the left and right.
- conductioncyl.m Displays radial temperature distribution in an infinite cylinder with zero initial temperature and temperature at $r = R$ suddenly elevated to T_0 at $t = 0$.
- heatedcyl.m Displays radial temperature distribution in an infinite cylinder of radius R that is heated by a uniform volumetric generation of heat q as in ohmic heating of an electrical conductor. The initial temperature is zero and the boundary temperature is zero,
- semiinfiniteslabstep.m Displays temperatures in semiinfinite medium (halfspace) when the temperature at $x = 0$ suddenly changes from 0 to T_0 at $t = 0$ with the halfspace is initially at zero temperature.
- semiinfiniteslab.m Displays steady state sinusoidal temperatures in a semi-infinite slab when the surface ($x = 0$) temperature varies sinusoidally.
- conductionsphere.m Displays the thermal response of a homogeneous sphere driven from zero initial temperature by a sudden temperature change T_0 at the surface $r = R$.
- conductionspheresinusoid.m Displays the steady-state thermal response of a sphere to a sinusoidal temperature variation of amplitude T_0 at the surface $r = R$. The input is dimensionless sinusoid frequency $\omega R^2/\kappa$ where κ is the diffusivity of the material of the sphere.

conductionspherecooling.m Displays the thermal response of a homogeneous sphere driven from initial temperature T_0 by a sudden temperature change to zero at the surface at $r = R$.

Beach Nourishment:

beachnourishment.m Displays the solution to the diffusion equation for an infinite domain with an initial rectangular beach projection planform.

String Vibration:

stringanimation.m Displays the d'Alembert solution to the plucked string problem. Input is the nondimensional location of the pluck, a/L .

stringvibration.m Displays the Fourier series solution to the plucked string problem. Input is the the nondimensional location of the pluck, a/L .

displacementexcitedstring.m Displays standing waves in a taut string fixed at the right end and with sinusoidal motion of amplitude Y_0 at the left end. Input is the ratio of the frequency of excitation to the first natural frequency of the string.

forcedstring.m Displays the motion of a taut string forced at $x = a$ by a sinusoidal force with amplitude P_0 . Inputs are the ratio of the forcing frequency to the first natural frequency of the string and the nondimensional location of the force, a/L .

Groundwater Drawdown:

gndw1.m Solves two layer aquifer problem and stores the solution in a data file framedata.mat to be played back by gndw2.m (Jacob model).

framedata.mat Contains data file generated by the gndw1.m script to be played back in gndw2.m.

gndw2.m Displays data generated in gndw1.m which is loaded as an array saved in framedata.mat.

gndw3.m Solves and displays the solution to the one layer aquifer (Theis model).

Wave Propagation in Elastic Bars

elasticbar.m Displays the solution to an elastic bar fixed on the left end and free on the right end with zero initial velocity and an initial linear displacement field (constant strain) at $t = 0$.

elasticbar2.m Displays the solution to an elastic bar fixed on the left end with a suddenly applied constant force F on the right end and no initial velocity or displacement.

elasticbar4.m Displays the solution to an elastic bar fixed at the left end, free at the right end, driven by a compressive impulse of intensity I_0 at the free end. The bar has no initial deformation or velocity.

torsionbar.m Displays torsional wave propagation in a round bar with an initial twist proportional to the distance from the fixed end and no initial velocity.

torsionbar2.m Displays torsional wave propagation in a round bar fixed at the left end with a suddenly applied constant torque T to the right end and no initial angular twist or velocity.

Static Beam Bending Problems

ssbeamoneload.m Displays the static shear, bending moment and deflection for a simply-supported beam as a load P traverses from left to right.

ssbeamtwo loads.m Displays the static shear, bending moment and deflection as two loads of value P traverse a simply-supported beam. The loads are 30% of the span length apart in spacing.

cantileverbeamoneload.m Displays the shear, bending moment and deflection as a load P traverses a cantilever beam from the fixed end to the free end.

Fluid Dynamics

Blasius.m Displays the flowfield for the viscous, incompressible flow over a flat plate by first solving the Blasius equation. The solutions for the velocity field come from the solution to the Blasius equation and the derivatives thereof.

General Mathematics

centrallimit.m Displays the probability density functions (from a histogram) for the sum of n random uniformly distributed variables where n varies from zero to 20. At each step the sum is scaled so as to have zero mean and unity variance.

GibbsPhenom.m Animates the evolution of the Gibbs phenomenon for a Fourier series representation of a square wave by continuously summing the terms and displaying the resulting waveform for 61 terms.